

Asymmetric All-Pay Contests with Spillovers*

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Abstract

When opposing parties compete for a prize, the sunk effort players exert during the conflict can affect the value of the winner's reward. These *spillovers* can have substantial influence on the equilibrium behavior of participants in applications such as lobbying, warfare, labor tournaments, marketing, and R&D races. To understand this influence, we study a general class of asymmetric, two-player all-pay contests where we allow for spillovers in each player's reward. The link between participants' efforts and rewards yields novel effects – in particular, players with higher costs and lower values than their opponent sometimes extract larger payoffs.

Keywords: all-pay, contests, auctions, spillovers, war of attrition.

JEL Codes: C65, C72, D44, D62, D74

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1 Introduction

All-pay contests model strategic interaction among players who must expend some non-refundable effort in order to win a prize. They have been applied in diverse settings such as labor (Rosen 1986), R&D races (Che and Gale 1998; Dasgupta 1986), and litigation (Baye et al. 2005). For tractability, the recent literature mostly assumes that players' actions affect their opponent's probability of winning, but not the value of the prize. Yet, in many settings, such *spillover effects* on the prizes themselves arise naturally.

For example, consider the setting in Che and Gale 1998, where two lobbyists compete in an all-pay auction to win an incumbent politician's favor through campaign contributions. If the politician were instead a candidate running for office, then she would only be able to provide the reward if successfully elected. In this case, it is natural to assume that total campaign contributions increase the candidate's chances of prevailing. Therefore, each lobbyist's contributions increase her opponent's value for winning the politician's political favor. This raises new questions: is it better to curb one's own contributions to make their opponent lose interest? Or is it preferable to ramp up the competition? These questions have been largely left unanswered.

In other settings, spillovers may be designed. Consider an all-pay version of a standard labor tournament, in which division managers apply effort towards some production technology in order to win a promotion awarded to the most productive division. To maximize aggregate effort, a principal might choose to make the value of this promotion depend on everyone's performance in the contest. For example, if the promotion is for a partnership or involves stock options, the prize will be increasing in the efforts of all players. The effect that such compensation schemes have on the equilibrium has not yet been studied.

This paper identifies the equilibrium strategies and payoffs in general two-player contests with spillovers and establishes their uniqueness. We consider games with (i) complete information, (ii) deterministic prizes, (iii) at least partially sunk investment costs, and (iv) a general dependence of each participant's value for the prize on both players' actions. This family of games includes all-pay contests (Siegel 2009, 2010), while also allowing for general spillovers to affect the winner's payoff.¹ These games

¹We also allow the payoffs of the losers to depend separably on the actions of other players. See Section 8.

also overlap with the symmetric linear contests with spillovers studied in Baye et al. 2012. Unlike that paper, we restrict attention to all-pay contests, but allow for asymmetric equilibria and asymmetric, nonlinear payoffs.²

The addition of spillovers can have a significant impact on equilibrium behavior. First, players with strictly higher costs can have higher payoffs than those with lower costs, even if their value functions for the prize are identical. In fact, in some settings, players could increase their payoffs if they were allowed to commit to a schedule of costly handicaps (See Section 4). Thus, trying to favor an “underdog” participant in a contest by means of reducing their costs may have the opposite of the desired effect, and in fact decrease their welfare in equilibrium. This is also important in settings in which players can commit to increasing their costs (e.g. by selecting an inefficient technology), as they may choose to do so.

Another contribution of this paper is a novel procedure to construct equilibrium strategy profiles. The equilibrium strategy distributions of asymmetric all-pay contests have two distinct parts: the densities and a mass point at zero. In the literature on all-pay contests without spillovers, starting with Baye et al. 1996, expected payoffs are obtained independently of the equilibrium distribution. This independence is exploited to derive the probability mass at zero for the weaker player from the payoffs, which is then used to compute the densities. In the presence of spillovers, however, a player’s payoffs cannot be derived without the equilibrium strategy of their opponent. Because of this, the same process cannot be followed. To overcome this difficulty, we introduce an algorithm that works in exactly the opposite order: first, it solves for the density independently of the mass point, and then uses this density to find the probability mass at zero.

Our method capitalizes on the theory of Volterra Integral Equations (VIEs), which are integral equations with a unique fixed-point that can be obtained via iteration. To the best of our knowledge, these techniques have not previously been applied to the determination of equilibrium mixed-strategy profiles.³

The game we study is general enough to encompass many different applications in

²We discuss our relation to the literature at the end of the paper. Even in the symmetric linear case, no previous paper has established equilibrium uniqueness for all-pay contests with spillovers.

³Few other works in Economics use VIE methods in general. We note McAfee, McMillan, et al. 1989 and McAfee and Reny 1992 as some early examples. More recently, Gomes and Sweeney 2014 also used VIEs, to compute the unique efficient equilibrium bidding functions in generalized second-price auctions.

which spillovers matter. In particular, investment wars, contests with winner’s regret, and militaristic conflicts all fit our framework, since spillovers are key in each of these settings. Our model also subsumes a natural extension to the war of attrition which, unlike the classical model, yields a unique equilibrium on a bounded support. We are also able to use the same framework to describe wars of attrition where rational agents face uncompromising (never-yielding) types with positive probability, as in Abreu and Gul 2000 and Kambe 2019. Our approach identifies why these games admit unique equilibria when the regular war of attrition does not: the addition of an uncompromising type introduces an unavoidable cost that depends on a player’s own score, and we show this single characteristic is sufficient for a unique equilibrium.

The paper is organized as follows. We introduce the model, the equilibrium concept and the assumptions in Section 2. We construct the equilibrium and prove its uniqueness in Section 3. Section 4 presents sufficient conditions under which a player has a positive expected payoff. This includes an example where a player with higher costs and lower values receives a positive expected payoff while her opponent receives zero. In Section 5, we move to applications, and introduce a general perturbation of the war of attrition that ensures the equilibrium is unique. This perturbation admits the war of attrition with the possibility of an uncompromising type as a special case. Section 6 explores three other applications of the model where the equilibrium can be simplified using results from the following Section 7. In Section 8, we extend the analysis to contests with: (1) linearly separable spillovers on the losers’ payoff and (2) more than two players with ranked costs. We show that when costs are ranked, only two players participate in equilibrium. In Section 9, we review the related literature and discuss the results.

2 Model

We focus, for now, on contests with two participants. Extensions with more than two players are considered in section 8.2.

An asymmetric contest with spillovers is a family $\{I, \{\tilde{S}_i\}_{i \in I}, \{u_i\}_{i \in I}\}$, where

1. $I := \{1, 2\}$ is the set of players.
2. For each $i \in I$, $\tilde{S}_i := [0, \infty)$ is Player i ’s action space,⁴ i.e., her set of available

⁴We use a tilde because a later assumption will allow us to replace the action set with a bounded

scores s_i . We let s_{-i}, \tilde{S}_{-i} denote the action and action space, respectively, of Player $j \neq i$.

3. For each $i \in I$, $u_i : \tilde{S} \rightarrow \mathbb{R}$ is Player i 's payoff, where $\tilde{S} := \prod_{i \in I} \tilde{S}_i$.

Let $s := (s_i; s_{-i})$ denote an arbitrary element of \tilde{S} . Then, for each $(s_i; s_{-i})$, we further define

$$u_i(s_i; s_{-i}) := p_i(s_i; s_{-i})v_i(s_i; s_{-i}) - c_i(s_i)$$

where (i) $p_i(s_i; s_{-i})$ denotes the probability that i wins the prize given the score profile $(s_i; s_{-i})$, with $p_i(s_i; s_{-i}) = 1 - p_{-i}(s_{-i}; s_i)$ and

$$\begin{aligned} p_i(s_i; s_{-i}) &= 1 && \text{if } s_i > s_{-i}, \\ p_i(s_i; s_{-i}) &= \lambda \in [0, 1] && \text{if } s_i = s_{-i}, \\ p_i(s_i; s_{-i}) &= 0 && \text{if } s_i < s_{-i}; \end{aligned}$$

(ii) $v_i : \tilde{S} \rightarrow \mathbb{R}_+$ maps each score profile $(s_i; s_{-i})$ to Player i 's value $v_i(s_i; s_{-i})$ from winning the prize, and (iii) $c_i : \tilde{S}_i \rightarrow \mathbb{R}_+$ outputs Player i 's private cost $c_i(s_i)$ given her submitted score s_i .

Definition 1 (Two-player contest with spillovers). A two-player contest is said to have *spillovers* if, for some $i \in I$ and $s_i \in \tilde{S}_i$, there exists $s_{-i}, \hat{s}_{-i} \in \tilde{S}_{-i}$ such that

$$v_i(s_i, s_{-i}) \neq v_i(s_i, \hat{s}_{-i})$$

i.e., the prize's value for at least one player and an action of that player is not constant in their opponent's action.

Accommodating spillovers is the distinguishing feature of our analysis. As is standard, we are interested in characterizing the Nash equilibrium of these general contests.

Definition 2 (Best-responses). Consider a two-player contest $\{I, \{\tilde{S}_i\}_{i \in I}, \{u_i\}_{i \in I}\}$. For each $i \in I$, let $\Delta\tilde{S}_i$ denote the set of probability distributions on \tilde{S}_i and let $\Delta\tilde{S} := \prod_{i \in I} \Delta\tilde{S}_i$. Player i 's best response set $b_i(G_{-i})$ to $G_{-i} \in \Delta\tilde{S}_{-i}$ is given by

$$b_i(G_{-i}) := \arg \max_{s \in \tilde{S}_i} \int_{\tilde{S}_{-i}} u_i(s; s_{-i}) dG_{-i}(s_{-i})$$

interval.

Definition 3 (Nash equilibrium). Consider the two-player contest $\{I, \{\tilde{S}_i\}_{i \in I}, \{u_i\}_{i \in I}\}$. A Nash equilibrium of this game is a profile $\mathbf{G}^* := (G_i^*)_{i \in I} \in \prod_{i \in I} (\Delta \tilde{S}_i)$ where, for each $i \in I$, $\text{supp}(G_i^*) \subseteq b_i(G_{-i}^*)$.

2.1 Assumptions

The following assumptions are imposed throughout whenever a two-player contest is invoked.

Assumption 1 (A1, Smoothness). *The function $v_i(s_i; y)$ is continuously differentiable in s_i and continuous in y for all $i \in I$, $s_i \in \tilde{S}_i$, and $y \in \tilde{S}_{-i}$ with $s_i \geq y$. The function $c_i(s_i)$ is continuously differentiable in s_i for all $i \in I$, $s_i \in \tilde{S}_i$.*

Assumption 2 (A2, Monotonicity). *For all $i \in I$ and $s_i > 0$, $c'_i(s_i) > 0$ and*

$$v'_i(s_i; y) < c'_i(s_i)$$

for almost all y , where $v'_i(s; y) := \frac{\partial v_i(s; y)}{\partial s_i}$.

Assumption 3 (A3, Interiority). *For all $i \in I$,*

$$v_i(0, 0) > c_i(0) = 0 \quad \text{and} \quad \limsup_{s_i \rightarrow \infty} \sup_{y \in \tilde{S}_{-i}} v_i(s_i; y) < \lim_{s_i \rightarrow \infty} c(s_i).$$

Versions of assumptions A1, A2, and A3 are adopted by all papers in the all-pay contests literature. A2 formalizes the sense in which these contests are all-pay⁵ and A3 ensures that bids are positive and bounded.

Note that, for each $i \in I$, there exist $T_i \in \tilde{S}_i$ such that Player i will never choose a score $s \geq T_i$. Thus, we can restrict the action space to $S_i := [0, T_i]$.

Assumption 4 (A4, Discontinuity at ties). *For all $i \in I$ and $s \in S_i \cap S_{-i}$,*

$$v_i(s; s) > 0.$$

Assumption A4 is a novel, yet natural assumption. It states that agents would prefer to win a tie than lose one. It is satisfied if the prize is always valuable (i.e.

⁵That is, bids are costly for both the winner and loser.

winning is better than losing). Note that this assumption is equivalent to assuming a discontinuity in payoffs at ties⁶ – a property of all-pay auctions.⁷

3 Characterization of equilibrium

By standard arguments,⁸ any pair of equilibrium strategies will be mixed with support on some interval $[0, \bar{s}]$ and at most one player will have a mass point at zero. Players must therefore be indifferent between all points on their interval support:

$$\bar{u}_i(G_{-i}) := \int_0^s v_i(s; y) dG_{-i}(y) - c_i(s) \quad \text{for all } s \in [0, \bar{s}]. \quad (1)$$

Any pair of distributions (G_1, G_2) that satisfy (1) is an equilibrium. This paper's main contribution to the literature is the characterization of the solution to this system of equations, and showing that it is unique.

Theorem 1. *Every two-player all-pay contest has a **unique Nash equilibrium** $(G_i^*)_{i \in I} \in \prod_{i \in I} (\Delta S_i)$ in mixed strategies. Furthermore,*

$$G_i^*(s) = \int_0^s \tilde{g}_i(y) dy + \int_{\bar{s}}^{\bar{s}_i} \tilde{g}_i(y) dy, \quad (2)$$

where $\tilde{g}_i(s)$ solves

$$\tilde{g}_i(s) = \frac{c'_{-i}(s)}{v_{-i}(s; s)} - \int_0^s \frac{v'_{-i}(s; y)}{v_{-i}(s; s)} \tilde{g}_i(y) dy, \quad (3)$$

\bar{s}_i solves $\int_0^{\bar{s}_i} \tilde{g}_i(y) dy = 1$ and $\bar{s} = \min_{i \in I} \bar{s}_i$.

We outline the proof here with an emphasis on the general methodology. We show in the appendix that in any equilibrium, players choose strictly increasing, continuous mixed strategies with common support on some interval $[0, \bar{s}]$, as in (1), and that at most one participant can have a mass point at zero. Moreover, differentiating (1) yields the equivalent condition (3), which therefore must be satisfied on $[0, \bar{s}]$ in equilibrium for some \bar{s} (Lemma 0 in Appendix).

⁶By A1 and A3, $v_i(s; s) \neq 0$ implies A4.

⁷In contrast, Tullock contests (Buchanan et al. 1980) and Lazear and Rosen contests (Lazear and Rosen 1981) are continuous at ties.

⁸These arguments can be referenced in the Appendix.

The key step is recognizing that we can apply results about Volterra Integral Equations (VIE) to show that (3) has a unique solution. The relevant result is summarized in the following Lemma. For a proof, see e.g. Brunner 2017.

Lemma 1 (Volterra 1896). *Let $K(s; y)$ and $f(s)$ be continuous functions. Then, the following integral equation*

$$g(s) = f(s) - \int_0^s K(s; y)g(y)dy \quad \text{for all } s \in [0, \bar{s}] \quad (4)$$

has a solution, g , unique almost everywhere. Moreover, (4) defines a contraction mapping; implying the solution can be found by iteration.

Note that (4) is the same as (3) for $f(s) := \frac{c'_i(s)}{v_{-i}(s;s)}$ and $K(s; y) := \frac{v'_{-i}(s;y)}{v_{-i}(s;s)}$. So, Lemma 1 implies that only one pair of functions $(\tilde{g}_1, \tilde{g}_2)$ solves (3). Next we show that the unique solutions are densities, i.e., for each i there is an interval $[0, \bar{s}_i]$ where \tilde{g}_i is non-negative and integrates to one.

Lemma 2. *Assume a two-player contest where $(\tilde{g}_i)_{i \in I}$ satisfy the indifference condition in (3). Then, for each $i \in I$, there exists $\bar{s}_i \in S_i$ such that*

$$\int_0^{\bar{s}_i} \tilde{g}_i(y)dy = \tilde{G}_i(\bar{s}_i) = 1, \quad (5)$$

and $\tilde{g}_i(s)$ is positive for $s \leq \bar{s}_i$.

Lemma 2 is proven in the appendix. We must now make the two densities have the same support. The next key insight is that there is exactly one way to do this. Recall that at most one player can have a mass point and that this mass point must be at zero (Lemma 0). If $\bar{s}_1 = \bar{s}_2$, then there is a unique equilibrium without any mass point. Otherwise, order the players such that $\bar{s}_1 < \bar{s}_2$. Then, give Player 2 a mass point of size $1 - \tilde{G}_2(\bar{s}_1)$. By construction, both players' densities integrate to one on the common support $[0, \bar{s}_1]$.

The above can be performed via the following algorithm:

1. Find each $\tilde{g}_i(s)$.
2. Integrate each $\tilde{g}_i(s)$ to find \bar{s}_i given by equation 5.

3. Take $\bar{s} = \min_i \bar{s}_i$ and give each player an mass point at zero of size

$$1 - \tilde{G}_i(\bar{s}),$$

which is positive for at most one player.

The three steps of the algorithm are illustrated by Figure 1.⁹

Since the cumulative distribution functions are useful, we sometimes use the alternate expression in Corollary 1.1.

Corollary 1.1. *Consider a two-player all-pay contest where $v_i(s; y)$ is continuously differentiable in both arguments for all $i \in I$.¹⁰ Then, we can alternatively express the unique equilibrium as*

$$G_i(s) = \left[\tilde{G}_i(\bar{s}_i) - \tilde{G}_i(\bar{s}) \right] + \tilde{G}_i(s),$$

where

$$\tilde{G}_i(s) = \frac{c_{-i}(s)}{v_{-i}(s; s)} + \int_0^s \frac{\partial v_{-i}(s; y)}{\partial y} \frac{\tilde{G}_i(y)}{v_{-i}(s; s)} dy. \quad (6)$$

4 Payoffs

Since payoffs are constant on the interval $[0, \bar{s}]$, each player i receives an expected payoff of $v_i(0; 0)G_{-i}(0) \geq 0$. Only one player can have an mass point (at zero), so there can be at most one player – their opponent – with a positive payoff.¹¹

In contests without spillovers, it is easy to identify the player with a positive payoff when normalized costs (i.e. the cost-value ratio) are ranked. That is, if

$$\frac{c_i(s)}{v_i(s)} < \frac{c_{-i}(s)}{v_{-i}(s)} \quad (7)$$

for all $s > 0$, then player i has a positive payoff.

This is because, by our algorithm, Player i 's opponent has an mass point if and only if $\tilde{G}_i(\bar{s}) > \tilde{G}_{-i}(\bar{s})$. Applying Corollary 1.1, yields the equivalent condition:

⁹Figure is for an example contest with $v_1(s_1; s_2) = 2 + s_1 + 2s_2$, $v_2(s_2; s_1) = 1 + s_2 + 2s_1$, $c_1(s_i) = 3s_1$, and $c_2(s_2) = 4s_2$.

¹⁰A1 only imposes that $v(s; y)$ is continuous in both arguments and continuously differentiable in the first argument.

¹¹In a symmetric contest, note that both players receive an expected payoff of zero.

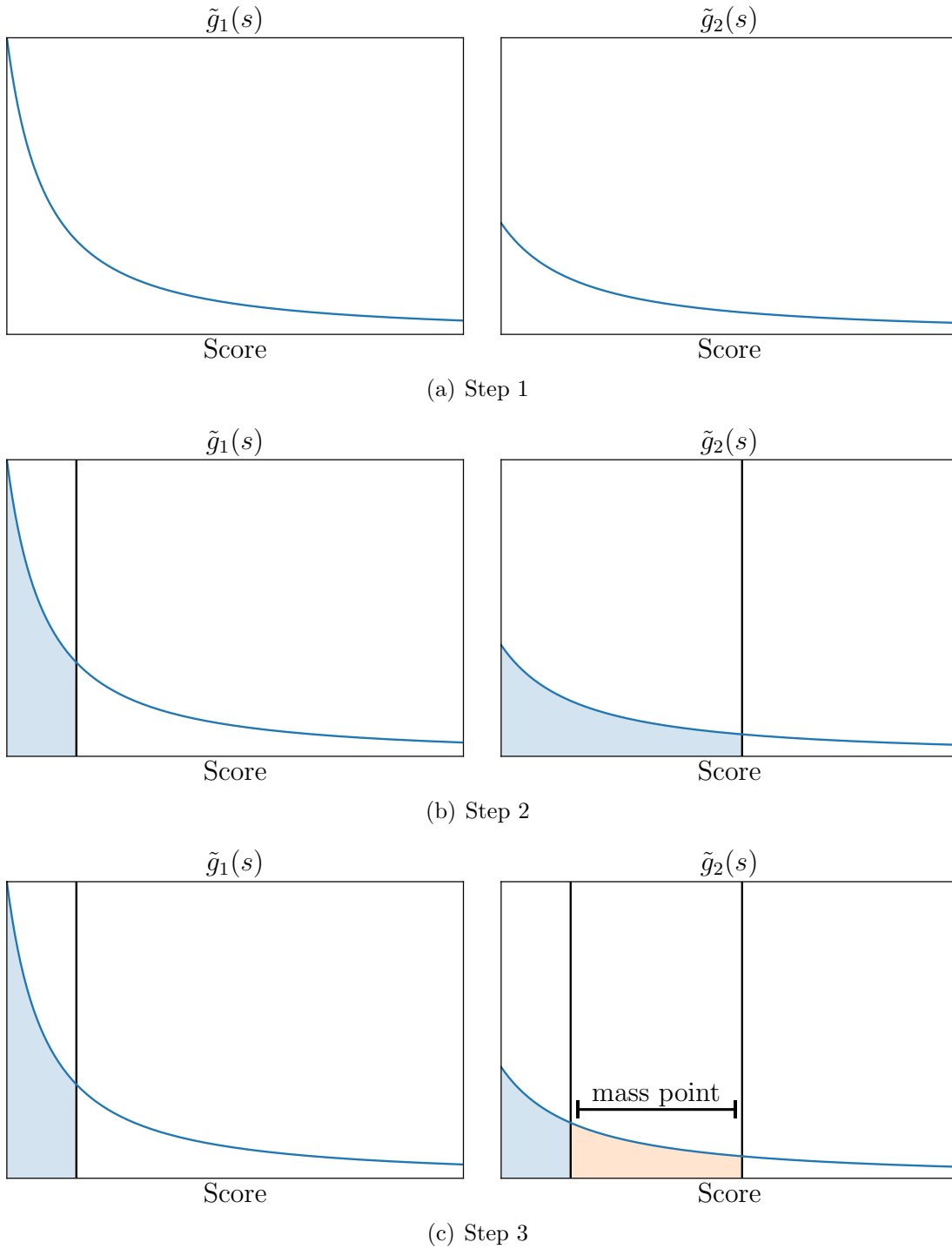


Figure 1: The three steps of the algorithm for finding the equilibrium strategies g_1 and g_2 . Begin with \tilde{g}_1 and \tilde{g}_2 (Step 1); find the cutoff points where each \tilde{g}_i integrates to 1 (Step 2). Finally, enforce identical supports if necessary, by transferring the excess probability mass of at most one player to zero (Step 3).

$$\frac{c_i(\bar{s})}{v_i(\bar{s}; \bar{s})} + \int_0^{\bar{s}} \frac{\partial v_i(\bar{s}; y)}{\partial y} \frac{\tilde{G}_{-i}(y)}{v_i(\bar{s}; \bar{s})} dy < \frac{c_{-i}(\bar{s})}{v_{-i}(\bar{s}; \bar{s})} + \int_0^{\bar{s}} \frac{\partial v_{-i}(\bar{s}; y)}{\partial y} \frac{\tilde{G}_i(y)}{v_{-i}(\bar{s}; \bar{s})} dy$$

In the absence of spillovers, the integral terms are equal to zero and the condition is implied by (7). In the presence of spillovers, one must impose a condition on the integrals.

Theorem 2. *Consider a two-player contest where $v_i(s; y)$ is continuously differentiable in s and y for all $i \in I$. Suppose that the following two conditions hold:*

$$\frac{c_i(s)}{v_i(s; s)} < \frac{c_{-i}(s)}{v_{-i}(s; s)} \quad (8)$$

$$\frac{1}{v_i(s; s)} \left| \frac{\partial v_i(s; y)}{\partial y} \right| \leq \frac{1}{v_{-i}(s; s)} \frac{\partial v_{-i}(s; y)}{\partial y} \quad (9)$$

for all $s \in (0, \bar{s}]$ and $y \in [0, s]$. Then, Player i has a positive payoff.

Theorem 2 gives an analogue of (7) for some contests with spillovers. The proof is in the Appendix. Condition (8) is the same as (7), while Condition (9) additionally imposes two extra requirements: (1) spillovers increase the value of the prize¹² and (2) player i is less dependant on these spillovers than her opponent.

Theorem 2 cannot be applied when spillovers decrease the value of the prize for both players. In this case, as demonstrated in Example 1, a player with strictly higher costs can receive a positive payoff even if both players have the same value function v for the prize.

Example 1 (Higher cost player has positive payoffs). Consider a two-player contest with spillovers. Let $c_1(s) = s^2$, $c_2(s) = s$, and $v(s; y) := v_1(s; y) = v_2(s; y)$ be given by:

$$v(s; y) = \frac{2}{5} + \frac{1}{1 + e^{\lambda(2y-1)}}$$

where $\lambda \geq 0$ is an exogenous parameter that determines the size of the spillovers. For all $y > 0$, v is strictly decreasing in y with $v(s; y) < 0.9$. So, the support of the players strategies is contained in $[0, 0.9]$. On this interval, Player 1 has a strict cost advantage.

¹²The condition only imposes this for Player $-i$.

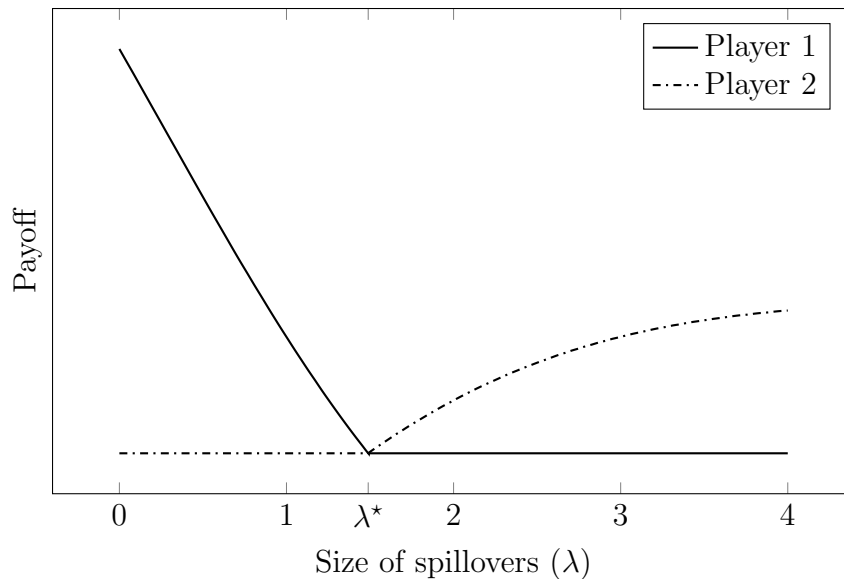


Figure 2: Player 2 has strictly higher costs. However Player 2 receives a positive payoff when spillovers are sufficiently large ($\lambda > \lambda^* \approx 1.489$).

When $\lambda = 0$, the prize is constant such that $v(s; y) = 0.9$. In this case, Player 1 receives a positive payoff. When we increase λ , this payoff decreases until we reach some $\lambda^* \approx 1.489$ such that both players receive a payoff of zero. For all $\lambda > \lambda^*$, Player 2 receives a positive payoff despite having strictly higher costs. The payoffs of both players are plotted in Figure 2.

△

The reversal in Example 1 occurs because marginal costs are not ranked. While Player 1 has lower costs in absolute terms, Player 2 has a lower marginal cost for all scores above $\frac{1}{2}$. This causes Player 2 to place comparatively more density on these bids. As can be seen in Figure 3, spillovers make the prize sharply less valuable when the opponent bids above $\frac{1}{2}$. So, these higher bids from player 2 damage player 1's valuation enough to reduce her participation.

Example 1 highlights a potential problem when giving one side an advantage in a contest. In the presence of spillovers, decreasing a player's costs can reduce their welfare in equilibrium. The example also implies that it's possible to have a contest where one or more players would prefer to ex-ante increase their own costs.¹³

¹³Suppose both players are as in Example 1 except $c_1(s) = c_2(s) = s^2$. Then, the game is symmetric. So, both players have a payoff of zero. If player i increased her cost to $c_i(s) = s$, then she would receive a positive expected payoff, as in the example.

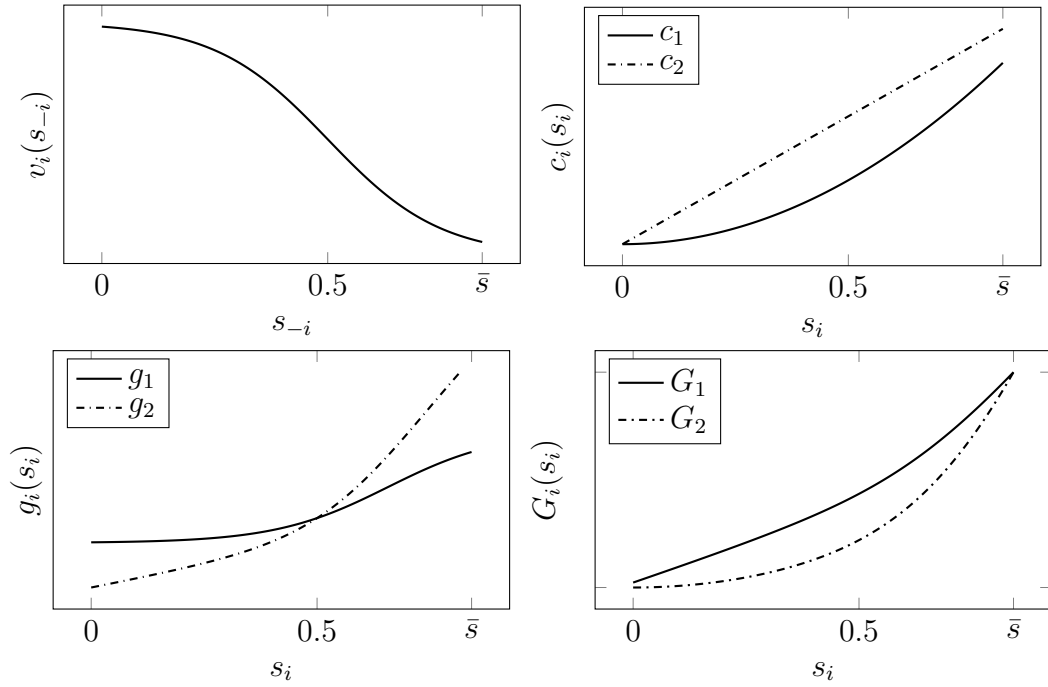


Figure 3: Value (upper left), costs (upper right), strategy densities (lower left), and distributions (lower right) for Example 1 with $\lambda = 4$. Note that Player 2 has a lower marginal cost for scores above $\frac{1}{2}$ and, because of spillovers, these scores devalue the prize for Player 1.

5 War of attrition with costly preparation

To illustrate the wide applicability of our framework, we first explore extensions to the classical war of attrition (henceforth WoA). The WoA first appeared in theoretical biology to explain how individual selection works in favor of animal species that outlast others (Smith 1974). In Economics, WoAs have since been popularized in the study of bargaining (Abreu and Gul 2000; Kambe 2019), filibusters (Bulow and Klemperer 1999), delays in the implementation of stabilizing macroeconomic policies (Alesina and Drazen 1991) exit, competition and price wars (Fudenberg and Tirole 1986; Ghemawat and Nalebuff 1985), boycotts and activism (Egorov and Harstad 2017), among others.

The canonical war of attrition is a game between two players $i = 1, 2$. Each picks a score¹⁴ in $[0, \infty)$ and the player i to select the largest score s_i wins an amount that depends on the loser's choice s_{-i} .¹⁵ A player's payoff function is thus given by:

$$u_i(s_i; s_{-i}) = \begin{cases} f_i(s_{-i}) & \text{if } s_i > s_{-i} \\ \ell_i(s_i) & \text{if } s_i < s_{-i} \\ \alpha_i f_i(s_{-i}) + (1 - \alpha_i) \ell_i(s_i) & \text{if } s_i = s_{-i} \end{cases}$$

where f_i, ℓ_i are strictly decreasing, continuously differentiable functions such that $f_i(s) > \ell_i(s)$, $\lim_{s \rightarrow \infty} \ell_i(s) = -\infty$, $\ell_i(0) = 0$, and $\alpha_i = 1 - \alpha_{-i} \in (0, 1)$.

The typical WoA is known to admit multiple equilibria and therefore does not satisfy the assumptions in Section 2.1.¹⁶ We propose a general perturbation that does. Suppose, the winner's outcome is decreasing in her own score – even if this dependence is minimal:

$$u_i(s_i; s_{-i}) = \begin{cases} f_i(s_{-i}) - \varepsilon_i(s_i) & \text{if } s_i > s_{-i} \\ \ell_i(s_i) - \varepsilon_i(s_i) & \text{if } s_i < s_{-i} \\ \alpha_i f_i(s_{-i}) + (1 - \alpha_i) \ell_i(s_i) - \varepsilon_i(s_i) & \text{if } s_i = s_{-i} \end{cases}$$

for any strictly increasing continuously differentiable function ε_i with $\varepsilon_i(0) = 0$ and

¹⁴In the classic war of attrition, in general each player will chose an exit time.

¹⁵Again in the classic war of attrition, this value is typically the fixed value of a prize, time-discounted for however much the losing opponent lasted in the game.

¹⁶In particular, it violates assumptions A2 and A3.

$\lim_{s \rightarrow \infty} \varepsilon_i(s) > f_i(0)$ for all i .

We denominate this variant a *WoA with costly preparation*, as there is some small preparation cost $\varepsilon(s)$ incurred to set score s – i.e. the maximum amount of time s one wishes to participate for. For example, a company engaged in a price war might have to build up inventory in advance or secure a costly line of credit.

Alternatively, We could also call this a WoA with imperfect monitoring. Indeed, the costly preparation is equivalent to a small probability of not detecting that the war of attrition is over. According to this interpretation, you determine an exit point in advance and exit early only if you are aware that your opponent has exited.

A WoA with costly preparation/imperfect monitoring fits the two-player all-pay contest with spillovers where

$$v_i(s_i; s_{-i}) := f_i(s_{-i}) - \ell_i(s_i) \text{ and}$$

$$c_i(s_i) := \varepsilon_i(s_i) - \ell_i(s_i),$$

which satisfy assumptions A1-4. Therefore, this game has a unique equilibrium, and there exists some \bar{s} such that no player bids above \bar{s} . Theorem 1 further allows us to characterize the equilibrium and Proposition 1 gives a closed form expression for the equilibrium strategies.

As the preparation costs become small (with $\varepsilon'_i(s) \rightarrow 0$ uniformly for all s), the unique equilibrium of a WoA with costly preparation approaches the mixed-strategy equilibrium of the classic WoA.¹⁷

One of the counter-intuitive results from the standard WoA with fixed prize values is the fact that both players get expected payoffs of zero, and the player with the lowest value wins most of the time. Example 2 shows why the higher-value player can receive a positive payoff in our perturbation.¹⁸

Example 2 (WoA with costly preparation). Consider a WoA where $f_1(t_2) = 1 - t_2$, $f_2(t_1) = 2 - t_1$, and $\ell_i(t_i) = -t_i$ for each $i \in \{1, 2\}$. The equilibrium of this game is:

$$G_1(s) = 1 - \exp\left(-\frac{s}{2}\right) \quad G_2(s) = 1 - \exp\left(-\frac{s}{1}\right).$$

Note that Player 1's strategy first order stochastically dominates Player 2's. In fact,

¹⁷Refer to the Appendix for a direct proof.

¹⁸This is not always the case in the WoA with costly preparation. In fact, Example 1 is a WoA with costly preparation where v_i is constant.

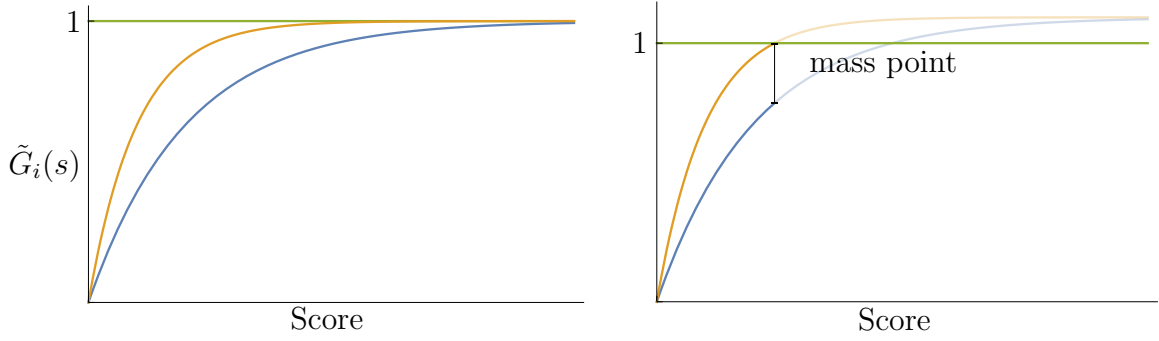


Figure 4: WoA (left) and WoA with costly preparation (right). If \tilde{G}_1 first-order stochastically dominates \tilde{G}_2 , our algorithm implies that Player 1 has an atom. However, when the support is unbounded, there need not be an mass point and the first-order stochastic dominance holds on the actual strategies.

Player 1 exits at half the rate of Player 2, and wins $\frac{2}{3}$ of the time – despite having the lower valuation.

In the corresponding WoA with costly preparation where $\varepsilon_i(s) := \delta s$, with $\delta > 0$, the equilibrium is

$$G_1(s) = (1 + \delta) \left(1 - \exp\left(-\frac{s}{2}\right) \right) + \underbrace{\sqrt{\delta^2 + \delta} - \delta}_{\text{mass point}}$$

$$G_2(s) = (1 + \delta) \left(1 - \exp\left(-\frac{s}{1}\right) \right).$$

Because Player 1 has a mass point, Player 2 receives a positive payoff. This is because the algorithm for finding the equilibrium of an all-pay contest stops at some finite \bar{s} . There, $\tilde{G}_2(\bar{s}) > \tilde{G}_1(\bar{s})$, implies that 2 has a positive payoff. Figure 4 provides a visualization on how the stronger player gets a positive payoff when the support is finite.

When δ is large enough ($\delta > \frac{2\sqrt{7}-5}{6} \approx 0.05$) Player 2 wins most of the time, in addition to receiving a positive payoff. \triangle

The WoA with costly preparation generalizes other perturbations that have a unique equilibrium. For example, Abreu and Gul 2000 and Kambe 2019 extend the WoA to let a rational player’s opponent be of an uncompromising type with positive probability, where “uncompromising” describes someone who bids (or exits at) infinity. Let z_i denote the (known) probability that player i is of an uncompro-

mising type.¹⁹ Against such an opponent, a rational or compromising player loses with certainty. This is a special case of the WoA with costly preparation where $\varepsilon_i(s) := -\frac{z_{-i}}{1-z_{-i}}\ell_i(s)$.

This relationship sheds light on the uniqueness of equilibrium found in the WoA with an uncompromising type. Indeed, by adding the possibility of a never-yielding opponent, we effectively introduce an unavoidable cost that depends on the player's own score. As was shown in the WoA with costly preparation, this characteristic is actually sufficient for a unique equilibrium.

6 Other applications

In this section we introduce several models that can be solved using the methods in this paper. Each model also illustrates a class of games with closed form solutions presented in Section 7.

6.1 Offensive/Defensive Balance

Military strategists generally agree that warfare is naturally asymmetric: the defending party can usually prevail with less expenditure of resources than the attacker (Clausewitz 1982). More generally, scholars have tried to identify which factors influence the so-called offensive/defensive balance – that is, the many elements of military technology that generate either offensive or defensive advantages, and thus affect the probability of war (Levy 1984). Our model is able to capture both the defensive advantage and the role of the prize-depleting nature of war in the offensive/defensive balance debate.

An attacker (a) invades a defender's (d) territory, which is worth V . Both combatants purchase costly scores in $[0, \infty)$, and the combatant with the higher score wins. A score of s_i costs $c_i s_i$, where $c_i > 0$ is a positive constant, for player $i \in I := \{a, d\}$. Furthermore, a 's score inflicts $\delta_a s_a$ damage to the territory.²⁰ If the attacker wins, it internalizes all costs faced by the defender, as these costs effectively depleted the resources available from the territory. Consider the following payoff functions

¹⁹Note whether or not a player i 's opponent of the uncompromising type is the only unknown information to i .

²⁰Assuming the defender also inflicts a cost of $\delta_d s_d$ onto the attacker does not change the analysis.

$u_a : [0, \infty) \rightarrow \mathbb{R}$ for the attacker:

$$u_a(s_a, s_d) = p_a(s_a, s_d)(V - \delta_a s_a) - c_a s_a,$$

and the following payoff function $u_d : [0, \infty) \rightarrow \mathbb{R}$ for the defender:

$$u_d(s_a, s_d) = (1 - p_a(s_a, s_d))(V - \delta_a s_a) - c_d s_d,$$

where $p_a(\cdot) : [0, \infty)^2 \rightarrow [0, 1]$ denotes the probability that the attacker is victorious. Accordingly, we let $p_a(s_a, s_d) = 1$ whenever $s_a > s_d$, $p_a(s_a, s_d) = 0$ when $s_a < s_d$, and $p_a(s_a, s_d) = \lambda \in [0, 1]$ whenever $s_a = s_d$.

When we transform this model into our framework, we get $c_i(s_i) := c_i s_i$ and

$$v_a(s_a; s_d) = v_d(s_d; s_a) := V - \delta_a s_a.$$

Assume it costs weakly more to attack than to defend (i.e., $c_a \geq c_d$). The attacker does not have any spillovers while the defender is harmed by her opponent.

We are able to leverage the linearity of payoffs in this case to obtain a closed-form solution to the problem.²¹ The defender receives positive payoffs if, and only if,

$$\bar{s}_d = \frac{V}{c_a + \delta_a} < \frac{V}{\delta_a} \left[1 - \exp\left(-\frac{\delta_a}{c_d}\right) \right] = \bar{s}_a,$$

which holds whenever $\delta_a > 0$ and $c_a \geq c_d$. In this case,

$$G_a(s) = 1 + \frac{c_d}{\delta_a} \log \left[\frac{c_a V}{(c_a + \delta_a)(V - \delta_a s)} \right] \quad \text{and} \quad G_d(s) = \frac{c_a s}{V - \delta_a s}.$$

The probability $P(s_a > s_d | \delta_a, c_a, c_d)$ that the attacker succeeds, in equilibrium, is given by

$$P(s_a > s_d | \delta_a, c_a, c_d) = \frac{c_d}{\delta_a^2} \left(\delta_a + c_a \log \left[\frac{c_a}{c_a + \delta_a} \right] \right) < \frac{c_d}{2c_a} \leq \frac{1}{2},$$

where the supremum is reached as $\delta_a \rightarrow 0$. If the war damages the territory at least as much as it costs the attacker to inflict such damage, ($\delta_a \geq c_a$), a tighter bound is obtained:

$$P(s_a > s_d | \delta_a, c_a, c_d) < 1 - \log(2) < \frac{1}{3}.$$

²¹Using Proposition 1, stated and proven in the next section.

Even if $c_a = c_d$, the defender is more than twice as likely to win than the attacker is. In our model, the stronger position of the defensive party comes as a byproduct of the inverse relationship between the attacker's strength and the erosion of the prize's value. This provides an alternate explanation on why it is typically easier to defend than to attack, something usually attributed to the high costs of maintaining long supply lines and of keeping seized territories (Glaser and Kaufmann 1998). The defender's stronger position also suggests that any positive participation cost in a war contest imposed on the aggressor would be effective in discouraging aggression.²²

6.2 War of Investment

Investment has long been considered as a method of committing to entry deterrence (Dixit 1980), while the war of attrition is a popular model of exit (Fudenberg and Tirole 1986). Our model can combine the two attributes into a single model of competition in continuous time, where players invest to stay in the game, but are able to recoup part of that investment if their opponent invests less.²³ Wars of investment can also be used to model Cold-War style defense spending and competition between technology companies and R&D races.

Assume two competitors, 1 and 2, invest in capital s_i at cost $c_i(s_i)$. The capital is necessary to engage in competition and depreciates at a constant rate. Competition results in zero profits. However, the winner is able to extract monopoly profits and benefits from the remaining capital according to an increasing function $v_i(s_i - s_{-i})$. More concretely, assume payoffs are

$$u(s_i; s_{-i}) = \begin{cases} v_i(s_i - s_{-i}) - c_i(s_i) & \text{if } s_i > s_{-i} \\ -c_i(s_i) & \text{if } s_i < s_{-i} \\ \alpha_i v_i(0) - c_i(s_i) & \text{if } s_i = s_{-i} \end{cases}$$

²²In the more general nonlinear model, where the value of the territory after invasion is given by $v_\delta(s_a)$ and the cost of choosing score s_i to player i is given by a continuously differentiable function $c_i : [0, \infty) \rightarrow \mathbb{R}_+$ satisfying the required assumptions A1 to A4, $c_d(s) \leq c_a(s)$ is sufficient to ensure that $\tilde{G}_a(s) < \tilde{G}_d(s)$ for all $s > 0$. By our algorithm, this guarantees the defender's payoff remains positive, with $G_d(s) = \frac{c_a(s)}{v_\delta(s)}$.

²³The combined model is similar to the war of attrition with costly preparation (Section 5). In fact, the two overlap when payoffs are linear. However, when payoffs are not linear, the two can be very different.

for any $\alpha_i \in [0, 1)$.

If assumptions A1-4 are met, there is a unique equilibrium of capital investments in mixed strategies on finite support. Moreover, the equilibrium admits a closed-form solution.²⁴

Example 3. Let $v_i(s; y) := e^{r_i(s-y)}\omega_i$ and $c_i(s) = e^{r_i s} - 1$, where $\omega_i, r_i \in (0, 1)$ for each $i \in I := \{1, 2\}$. Then,

$$\tilde{g}_i(s) = \frac{r_{-i}}{\omega_{-i}}$$

so the equilibrium strategies, excluding the possible mass point at zero, will be uniform with $\tilde{G}_i(s) = \left(\frac{r_{-i}}{\omega_{-i}}\right)s$.

The pair of ratios $\frac{\omega_i}{r_i}$ is therefore a sufficient statistic for the equilibrium of this game. Assume, without loss of generality that this ratio is weakly larger for Player 1. Then, the maximum duration of the game is Player 2's ratio $\bar{s} = \frac{\omega_2}{r_2}$.

The equilibrium is fully characterized by the overall *strength* of the players \bar{s} and the *competitive balance* $\delta := \frac{\omega_2/r_2}{\omega_1/r_1} \in (0, 1]$.

Because the strategies are uniform, Player 1's average commitment duration is half of the strength. Player 2 on the other hand has a mass point of size

$$G_2(0) = 1 - \delta$$

which decreases as the competition becomes more balanced.

Overall, the conflict is expected to last for

$$\mathbb{E}[\min(s_1, s_2)] = \int_0^{\bar{s}} (1 - G_1(y))(1 - G_2(y))dy = \frac{\delta\bar{s}}{3}$$

total periods. The relationship between overall power and war duration is one to one. The duration is also increasing in the competitive balance. So, a large strength differential implies the conflict will typically be short-lived, whereas close contests can have delayed resolutions.

△

²⁴The derivation of this closed-form solution uses Proposition 2 in the next section

6.3 All-pay contest with winner's regret

Winner's regret is the remorse that the winner has from spending more than is necessary to win a contest or auction. This phenomenon has mostly been studied in the context of winner-pay first-price, auctions (Engelbrecht-Wiggans 1989; Filiz-Ozbay and Ozbay 2007). We instead apply our framework to model winner's regret in an all-pay contest

Let each Player $i \in I := \{1, 2\}$ choose a score in $[0, \infty)$. Suppose i values the prize at $\mu_i(s_i)[1 - h_i(s_i - s_{-i})]$, where $\mu_i(s_i)$ is the player's objective value of the prize and $h_i(s_i - s_{-i})$ is the share of the winnings that is unappreciated due to regret. Each player pays the cost $c_i(s_i)$ whether they win or lose. So payoffs are

$$u(s_i; s_{-i}) = \begin{cases} \mu_i(s_i)[1 - h_i(s_i - s_{-i})] - c_i(s_i) & \text{if } s_i > s_{-i} \\ -c_i(s_i) & \text{if } s_i < s_{-i}, \\ \alpha_i \mu_i(s_i) - c_i(s_i) & \text{if } s_i = s_{-i}, \end{cases}$$

for any $\alpha_i \in [0, 1)$. We assume all functions are continuously differentiable with $c'(s) > 0$ and $h'_i(s) \geq 0$. Moreover, $\mu(0) > h(0) = c(0) = 0$ and $c'(s) > \mu'(s)$ for each s , so that lower bids are preferable even with no regret. Intuitively, the regret function, h , should not exceed one.²⁵ The equilibrium strategies will admit closed-form solutions.²⁶

Example 4. Let $\mu_i(s) := \omega_i \in (0, \frac{1}{2})$, $h_i(s) = \frac{s^2}{2}$ and $c_i(s) := s - \frac{s^2}{2}$ for $s \in [0, 1]$. Then,

$$\tilde{g}_i(s) = \frac{e^{-s}}{\omega_{-i}}.$$

Without loss of generality, let $\omega_1 \geq \omega_2$, implying Player 1 receives a non-negative payoff. Player 1 will thus play a truncated exponential distribution with parameter 1 and support $[0, -\log(1 - \omega_2)]$. Her expected score will be:

$$\mathbb{E}[s_1 | \omega_1, \omega_2] = 1 + \left(\frac{1 - \omega_2}{\omega_2} \right) \log(1 - \omega_2).$$

which depends negatively on her opponent's payoff scaling factor ω_2 . This is lower than in the same game without regret.

²⁵Note that this is not a technical requirement.

²⁶The derivation of this closed-form solution uses Proposition 3 in the next section.

The player with zero expected payoffs will place a mass point at zero of size:

$$G_2(0) = 1 - \frac{\omega_2}{\omega_1}$$

which is exactly the same size as if there were no regret. Player 2 will have expected score

$$\mathbb{E}[s_2|\omega_1, \omega_2] = \frac{\omega_2}{\omega_1} \left[1 + \left(\frac{1 - \omega_2}{\omega_2} \right) \log(1 - \omega_2) \right]$$

which is also less than in the same game without regret. The expected sum of the two scores score is:

$$\mathbb{E}[s_1 + s_2|\omega_1, \omega_2] = \left(1 + \frac{\omega_2}{\omega_1} \right) \left[1 + \left(\frac{1 - \omega_2}{\omega_2} \right) \log(1 - \omega_2) \right]$$

which is decreasing in ω_1 and increasing in ω_2 . In contests such as a labor tournaments, a large productivity differential between participants in the form of a high ω_1 and low ω_2 depresses aggregate effort. This is true in a contest with no spillovers, but the partial derivative of ω_1 is larger in absolute value when there is regret. That is, the effect is exacerbated by the fact that the stronger player is penalized for winning by a large margin. \triangle

7 Closed forms

In some cases, it is possible to express the equilibrium strategies in closed form. We consider classes of prize value functions where this is possible. 6 contains applications of each of the propositions below.

Proposition 1 (Linearly separable spillovers²⁷). *Consider a two-player contest where $v'_{-i}(s_{-i}; y)$ does not depend on y . That is, for each $i \in I$, $v_{-i}(s_{-i}; y) = v_{-i}^{-i}(s_{-i}) + v_{-i}^i(y)$. Then,*

$$\tilde{G}_i(s) = \frac{1}{f(s)} \int_0^s \frac{c'_{-i}(y)}{v_{-i}(y; y)} f(y) dy, \quad (10)$$

where $f(y) := \exp\left(-\int_0^y \frac{(v_{-i}^{-i})'(u)}{v_{-i}(u; u)} du\right)$.²⁸

²⁷For an application of this proposition, see the offensive/defensive balance example in Section 6.1.

²⁸Equation (10) defines the strategy of any player with no atom. For example, this is the case in the symmetric game.

When spillovers are not linear, we might still be able to find closed form solutions to equilibrium strategies. We highlight two particular cases where the VIEs in Equations (3) and (6) can be solved using Laplace transforms.

Definition 4 (Laplace Transform).²⁹ A function f defined on \mathbb{R}_+ admits a Laplace transform $F : \mathbb{C} \rightarrow \mathbb{C}$ given by

$$F(x) := \mathcal{L}\{f(s)\} = \int_0^\infty f(s)e^{-sx} ds$$

if and only if the above integral conditionally converges.³⁰

We require an extra technical assumption to ensure that the integral above converges. For simplicity, we will assume the relevant functions are of exponential order.

Definition 5 (Exponential order). A function f is of exponential order if and only if there exist $s', q, M \in [0, \infty)$ such that, for all $s \geq s'$,

$$|f(s)| \leq Me^{qs}.$$

Proposition 2 (Margin of victory spillovers). *Assume a two-player contest such that (i) for some i , $\frac{v'_i(s;y)}{v_i(s;s)} =: \nu_i(s-y)$ depends only on the score differential $s-y$, and (ii) ν_i and $\frac{c'_{-i}(s)}{v_{-i}(s;s)}$ are of exponential order.³¹ Then, for all $s \in (0, \bar{s}]$,*

$$\tilde{g}_{-i}(s) = \mathcal{L}^{-1} \left\{ \frac{\mathcal{L} \left\{ \frac{c'_i(s)}{v_i(s;s)} \right\}}{1 + \mathcal{L} \{ \nu_i(s) \}} \right\}$$

and

$$\tilde{G}_{-i}(s) = \mathcal{L}^{-1} \left\{ \frac{\mathcal{L} \left\{ \frac{c'_{-i}(s)}{v_{-i}(s;s)} \right\}}{x + x\mathcal{L} \{ \nu_i(s) \}} \right\},$$

where \mathcal{L} and \mathcal{L}^{-1} denote the Laplace and inverse Laplace transforms, respectively.

Proposition 2 can be used whenever the prize's value depends on the margin of victory, i.e. on the difference $(s-y)$ between the winning bid s and the losing bid y . We use Proposition 2 to solve the war of investment in Section 6.2.

²⁹See Churchill 1972 for an exposition on Laplace transforms.

³⁰That is, it does not need to converge absolutely.

³¹It is sufficient to assume that it admits a Laplace transform.

Proposition 3 (Multiplicative margin of victory spillovers). *Assume a two-player contest such (i) for some i , $\frac{1}{v_i(s;s)} \frac{\partial v_i(s;y)}{\partial y} =: \psi_i(s-y)$ depends only on the score differential $s-y$, and (ii) $\frac{\partial v_i(s;y)}{\partial y}$ and ψ_i and $\frac{c_i(s)}{v_i(s;s)}$ are of exponential order. Then, for all $s \in (0, \bar{s}]$,*

$$\tilde{g}_{-i}(s) = \mathcal{L}^{-1} \left\{ \frac{x \mathcal{L} \left\{ \frac{c_i(s)}{v_i(s;s)} \right\}}{1 - \mathcal{L} \{ \psi_i(s) \}} \right\}$$

and

$$\tilde{G}_{-i}(s) = \mathcal{L}^{-1} \left\{ \frac{\mathcal{L} \left\{ \frac{c_i(s)}{v_i(s;s)} \right\}}{1 - \mathcal{L} \{ \psi_i(s) \}} \right\}$$

where \mathcal{L} and \mathcal{L}^{-1} denote the Laplace and inverse Laplace transforms, respectively.

Proposition 3 can be used whenever the prize value is of the form $v_i(s;y) = v_i^1(s)v_i^2(s-y)$. For an application where we solve an all-pay contest with winner's regret, see Section 6.3.

8 Extensions

The first extension (8.1) allows for spillovers in the payoff of the loser. We show that this can be accommodated in our model so long as these spillovers are linearly separable. The second extension (8.2) allows for more than two players. We characterize an equilibrium with more players if the players' payoffs follow a particular ranked costs condition.

8.1 Separable spillovers on the losers payoff

There are several contexts where it makes more sense to have spillovers in the loser's payoff rather than the winner's. For example, models of litigation in English law must include the fact that the loser pays the winner's legal fees (Baye et al. 2005). While this may seem like a completely different scenario, it can easily be translated into our model if the cost is linearly separable. To see how, consider a two-player contest where player i 's payoff u_i is given by

$$u_i(s_i; s_{-i}) := p_i(s_i; s_{-i}) \hat{v}_i(s_i; s_{-i}) - (1 - p_i(s_i; s_{-i})) \left(c_i^\alpha(s_i) + c_i^\beta(s_{-i}) \right)$$

where $c_i^\alpha(s_i) : \tilde{S}_i \rightarrow \mathbb{R}_+$ is the portion of i 's costs that depends on their own score, and $c_i^\beta : \tilde{S}_{-i} \rightarrow \mathbb{R}_+$ is the portion of i 's costs that depends on $-i$'s scores.

The above fits our model once we let $v_i(s_i; s_{-i}) := \hat{v}_i(s_i; s_{-i}) + c_i^\beta(s_{-i}) + c_i^\alpha(s_i)$ and $c_i(s_i) := c_i^\alpha(s_i)$.³²

8.2 More players

In contests with spillovers and more than two players, many of the results considered here and in the existing literature are violated. Existence still holds (see Olszewski and Siegel 2019, for example), but uniqueness doesn't.

Even without spillovers, it is difficult to construct an equilibrium of a contest where the normalized costs are not ranked. However, when the normalized costs are ranked, only two players ever participate in equilibrium. This effectively collapses the problem into a two-player contest.

A generalized version of this condition holds in our setting where $n > 2$. We still require normalized costs to be ranked in some sense, but in a way that takes the spillovers into account.

Theorem 3. *Assume $i, j, i \neq j$, are two of the $n > 2$ players in a contest satisfying assumptions A1 to A4. Suppose that Player i has a positive payoff in the two-player contest where i and j are the participants, and that the following "ranked costs" condition holds for all $k \notin \{i, j\}$, $s \in \tilde{S}_k$, $s_i \in \tilde{S}_i$ and $s_j \in \tilde{S}_j$*

$$\frac{c_k(s)}{v_k(s; \mathbf{s}_{\{i,j\}})} \geq \frac{c_j(s)}{v_j(s; \mathbf{s}_{\{i\}})}, \quad (11)$$

where \mathbf{s}_H is a vector of opponent scores that is zero for all players not in set H . Then, there exists an equilibrium where only Players i and j participate.

To understand condition (11), consider the candidate equilibrium where Players

³²To see this, consider Player i 's expected utility in each model:

$$\begin{aligned} & \int_0^{s_i} v_i(s_i; y) dG_{-i}(y) - (1 - G_{-i}(s_i)) c_i^\alpha(s_i) - \int_{s_i}^{\bar{s}} c_i^\beta(y) dG_{-i}(y) && \text{for the original model} \\ & \int_0^{s_i} v_i(s_i; y) dG_{-i}(y) - (1 - G_{-i}(s_i)) c_i^\alpha(s_i) + \int_0^{s_i} c_i^\beta(y) dG_{-i}(y) && \text{in our framework} \end{aligned}$$

These two payoffs differ only by $\int_0^{\bar{s}} c_i^\beta(y) dG_{-i}(y)$, which is a constant. This insight is from Xiao 2018.

i and j compete using their two-player strategies and Player k does not participate. By not participating, Player k earns a payoff of zero – the same payoff as Player j . Condition (11) says that if she enters, Player k 's normalized cost will be higher at every point than Player j 's already is. Therefore, her payoff from participating is at most zero (Player j 's payoff). So, there is no profitable deviation for any player.

Note that it is possible for multiple interval equilibria to satisfy Theorem 3 when spillovers decrease the value of the prize. If this decrease is sufficiently large, it's reasonable to have $k \succ j$ and $j \succ k$ in the sense of (11).

In the absence of spillovers, multiple equilibria also arise with three or more players. However, if payoffs are asymmetric, there can be at most one equilibrium where the support of each player's strategy is a union of intervals. Additionally, the payoffs of each player are consistent across all equilibria. Neither of these properties hold in contests with spillovers. Indeed, payoffs can vary between different interval equilibria, as Example 5 illustrates..

Example 5. Suppose $n = 3$, and let:

$$\begin{aligned} v_i(s_i; s_{-i}) &= 1 - s_j - s_k, i = \{1, 3\}, \quad j, k \neq i \\ v_2(s_2; s_{-2}) &= \frac{3}{4} - s_1 - s_3 \end{aligned}$$

Further, assume all three players have identical cost functions $c_i(s_i) = s_i$ for all $i \in \{1, 2, 3\}$.

Consider now the following proposed equilibrium: Player 3 chooses $s_3 = 0$ with probability 1, while Players 1 and 2 submit scores as in the two-player equilibrium where only 1 and 2 participate, and thus choose

$$G_2(s) = \log\left(\frac{1}{3-4s}\right) + \log(4-e) \quad G_1(s) = \log\left(\frac{1}{1-s}\right).$$

We can show that this profile of strategies is indeed an equilibrium; Players 1 and 2 have no incentive to deviate. Moreover, there's no $s > 0$ such that Player 3 obtains a nonnegative payoff by playing s , given 1 and 2's distributions.

Now, clearly, Players 1 and 3 are identical, and so they are interchangeable. Thus, we could also have the following equilibrium: Player 1 chooses $s_1 = 0$ with probability 1, while

$$G_2(s) = \log\left(\frac{1}{3-4s}\right) + \log(4-e) \quad G_3(s) = \log\left(\frac{1}{1-s}\right)$$

Note that only Player 1 has a positive payoff in the first equilibrium and only Player 3 has a positive payoff in the second equilibrium. \triangle

9 Related literature and conclusions

Throughout this paper, we characterized and established uniqueness for the equilibrium of two-player contests using techniques from the theory of integral equations. This allowed us to derive insights on equilibrium payoffs, winners and losers, and on the importance of spillovers for applications. The fact that ranked normalized costs are not enough to establish dominance demonstrates how spillovers can favor high-cost, low-value players that nevertheless have a marginal cost advantage over their opponent when bids are high. In particular, the results in this paper suggest several potential consequences of legal structures, conflicts and competition.³³

This paper is most closely related to two others. Baye et al. 2012, also considers spillovers in two-player contests, but focuses on symmetric equilibria and linear symmetric costs and valuations. We show that there are no asymmetric equilibria and extend the analysis to include asymmetric players and general functional forms for the prize values. This allows us to establish equilibrium uniqueness, express novel results about payoffs, and characterize the equilibrium in different applications (Sections 5 and 6).

The second paper that approaches a similar question to our own is Xiao 2018. The author, however, focuses on constant prize value and separable spillovers in the cost functions, which are independent of winning or losing. This independence significantly restricts the equilibrium effects of the spillovers,³⁴ which is not true when spillovers are in the prize value. We applied Xiao's results to analyze contests with spillovers in the prizes and, linearly separable spillovers in the loser's payoff in Section 8.1.

³³This model does not require or imply that the results of a contest are known in advance. In fact, players are always uncertain of their own victory. However, this uncertainty stems from not knowing the resources that your opponent dedicated to the contest.

³⁴Linearly separable spillovers on the cost have no effect on the equilibrium while multiplicatively separable spillovers scale the cost of bids by an endogenous constant.

This paper is also connected more broadly to the literature of spillovers in other contest frameworks. Hodler and Yektaş 2012, for example, use a linear first-price auction with spillovers to model war.³⁵ Hirai and Szidarovszky 2013 and Damianov et al. 2018 consider Tullock contests where the value of the prize depends on the sum of the bids.

We identify several avenues for future work. The class of contests that include spillovers is very large and fits many applications. The fact that we are able to construct very different contests with the same equilibrium strategies (e.g. the all-pay contest with winner’s regret in Section 6.3 has the same equilibrium as a war of attrition with costly preparation of Section 5) suggest that it might be possible for a contest designer to induce behavior more cheaply through spillovers.

Other contest design problems where spillovers are available are also of interest. Appendix II contains a brief exposition that shows that when a constrained designer that cares about aggregate effort can reward contestants with prizes that may include spillovers, no contestant will be allowed positive rents. This in particular would make computing equilibrium strategies straightforward. Under what circumstances introducing spillovers is desirable to a contest designer is however still an open question.

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³⁵The authors refer to this as an all-pay contest, but only the winner actually pays.

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10 Appendix

Appendix I Proofs

Proof of Lemma 0

Lemma 0. (*Interval support*) *In any Nash equilibrium, players choose strictly increasing, continuous mixed strategies G_i with common support on some interval $[0, \bar{s}]$. At most one participant can have a mass point and it must be at zero.*

Moreover, the equilibrium is defined by the following indifference condition:

$$\tilde{g}_i(s) = \frac{c'_{-i}(s)}{v_{-i}(s; s)} - \int_0^s \frac{v'_{-i}(s; y)}{v_{-i}(s; s)} \tilde{g}_i(y) dy. \quad (3)$$

Proof. The argument is standard in the all-pay auction literature, and is presented here for completeness. Our proof is in several steps.

1. By A3, each player $i \in I$ would select scores in $S_i = [0, T_i]$, for T_i finite.
2. *The minimum score in the support of both players' strategies is zero.* Let $\underline{s}_1, \underline{s}_2$ denote the lower bounds of player 1 and player 2's strategies' supports, respectively. If $\underline{s}_1, \underline{s}_2 > 0$, then $\underline{s}_1 = \underline{s}_2$ – otherwise, player $i := \arg \min_{j \in I} \underline{s}_j$ would benefit from shifting the probability mass between $[\underline{s}_i, \underline{s}_{-i}]$ to zero. Let $\underline{s} := \underline{s}_1 = \underline{s}_2 > 0$. The two players can't both have a mass point at \underline{s} : either player would rather move that mass infinitesimally above \underline{s} . Similarly, it can't be that a single player i has a mass point at \underline{s} – then, they would benefit from shifting that mass to zero, as that would reduce costs without changing their probability of winning. If neither player places a mass point at \underline{s} , then there exists $\varepsilon > 0$ such that player i would prefer to shift the probability mass between \underline{s} and $\underline{s} + \varepsilon$ to zero.

Now, we also can't have $\underline{s}_i > \underline{s}_{-i} = 0$, as player i could reduce the lower bound of her support, reduce costs and leave her probability of winning unchanged.

3. *Both players will have the same maximum score in their strategies' support* (\bar{s}). Otherwise, the player with the highest upper bound to her support could reduce it and pay less costs without impacting the probability of winning.
4. *There are no gaps in the density.* If one player has a gap, then clearly both players must have the same gap. If two players have the same gap, then both players can benefit by moving some density from after the gap to just before.
5. *There are no mass points on the half open interval* $(0, \bar{s}]$. Otherwise, their opponent could ensure a positive payoff by shifting up density from below the mass point.
6. G_i^* *is strictly increasing in* $(0, \bar{s}]$. Suppose not, so that G_i^* is constant in some interval $(a, b) \subset (0, \bar{s}]$. Since there are no mass points in this interval, $G_i^*(a) = G_i^*(b) < 1$. Then, there exists $\varepsilon > 0$ such that i would benefit from shifting mass from $b + \varepsilon$ into a .
7. *At most one player will place a mass point at zero.* The two players can't both have a mass point at $\underline{s} = 0$: either player would rather move that mass infinitesimally above \underline{s} .

Because any such G_i^* is monotone increasing, it is differentiable almost everywhere.³⁶ Differentiating both sides of (2) then yields (3).

□

Proof of Lemma 2

Proof. The finite definite integral cannot diverge because the function is continuous. Also note that (3) gives us $g_i(0) = \frac{c'_{-i}(0)}{v_{-i}(0;0)} \geq 0$. This inequality is strict if $c'_{-i}(0) > 0$.

If the inequality is not strict, we find a positive value in a neighborhood of zero. Because c'_{-i} is positive everywhere but zero, we know that g_i is strictly increasing near zero. So there exists some $\delta > 0$ such that $g_i(s) > 0$ for $s \in (0, \delta)$.

We still need to confirm that $\tilde{g}_i(s) > 0$ on the relevant interval $\{s : \int_0^s |\tilde{g}_i(y)| dy \leq 1\}$. Suppose, by way of contradiction, that it is not. Then, by continuity, there must

³⁶In fact, G_i is continuously differentiable because both G_i and \tilde{g}_i are continuous.

be an initial point $s^* > 0$ such that $\tilde{g}_i(s^*) = 0$, $\int_0^{s^*} \tilde{g}_i(y)dy \leq 1$, and $\tilde{g}_i(s) \geq 0$ for all $s \leq s^*$. However, this is impossible because

$$\begin{aligned} \tilde{g}_i(s^*) &= \frac{1}{v_{-i}(s^*; s^*)} \left(c'_{-i}(s^*) - \int_0^{s^*} v'_{-i}(s^*, y) |\tilde{g}_i(y)| dy \right) \\ &\geq \frac{1}{v_{-i}(s^*; s^*)} \left[c'_{-i}(s^*) - \left| \max_{y \in [0, s^*]} v'_{-i}(s^*; y) \right| \underbrace{\left(\int_0^{s^*} |\tilde{g}_i(y)| dy \right)}_{\leq 1} \right] \\ &\geq \frac{1}{v_{-i}(s^*; s^*)} \underbrace{\left[c'_{-i}(s^*) - \left| \max_y v'_{-i}(s^*; y) \right| \right]}_{>0 \text{ (A2)}} > 0. \end{aligned}$$

We must now show that it is not possible for $\int_0^\infty |\tilde{g}_i(y)|dy \leq 1$. We can do this in one step with Holder's inequality.

$$c_{-i}(s) = \int_0^s v_{-i}(s; y) g_i(y) dy \leq \left(\int_0^s |g_i(y)| dy \right) \left(\max_{y \in [0, s]} v_{-i}(s; y) \right)$$

so $\int_0^s |g_i(y)| dy \geq \frac{c_{-i}(s)}{\max_y v_{-i}(s; y)}$ which is assumed to be greater than one as s approaches infinity (A3). By continuity, there exists an \bar{s}_i such that $\int_0^{\bar{s}_i} |g_i(y)| dy = 1$ (A1). \square

Proof of Corollary 1.1

Proof. Equation (6) is obtained by applying integration by parts to (1). This defines a Volterra Integral Equation which has a unique solution by lemma 1. This solution coincides with the one in Theorem 1 because Equation (1) cannot have two solutions. \square

Proof of Theorem 2

Proof. Consider equation (6). The main result of Beesack 1969 allows us to compare the solutions of two VIEs. In our setting, this means that conditions (8), (9) imply

$$\tilde{G}_2(s) \leq \tilde{G}_1(s) + \frac{c_1(s)}{v_1(s; s)} - \frac{c_2(s)}{v_2(s; s)} < \tilde{G}_1(s).$$

From this, it is clear that $\bar{s}_1 \leq \bar{s}_2$ which implies that player 2 has a mass point. The

bound comes from

$$u_1 = v_1(0; 0)(1 - \tilde{G}_2(\bar{s})) \geq v_1(0; 0) \left[\frac{c_2(\bar{s})}{v_2(\bar{s}; \bar{s})} - \frac{c_1(\bar{s})}{v_1(\bar{s}; \bar{s})} \right].$$

□

Proof that WoA with costly preparation approximates WoA

Proof. A direct application of (3) yields the following differential equation:

$$\tilde{g}_{-i}^\epsilon(s) = \frac{1}{\ell_i(s) - f_i(s)} \left(\ell_i'(s) \left[1 - \tilde{G}_{-i}^\epsilon(s) \right] + \epsilon'(s) \right)$$

Because this is a continuous linear mapping, we can take the limit as $\epsilon'(s)$ approaches zero. This simplifies to the same differential equation used to describe the equilibrium of the WoA (e.g. in Hendricks et al. 1988):

$$\frac{\tilde{g}_{-i}(s)}{1 - \tilde{G}_{-i}(s)} = \frac{\ell_i'(s)}{\ell_i(s) - f_i(s)}.$$

□

Appendix II Optimal Contest Design

In this section, we consider how a designer should bias a contest to increase the scores. Several papers have analyzed this problem of assigning prizes to maximize total scores, or the average score of the winner. For example, Mealem and Nitzan 2014 consider prize redistribution in a two-player all-pay contest with fixed values and symmetric costs. They show equalizing the prize values maximizes the total scores and that this contest yields weakly more total score than any similar Tullock-type lottery contest. Che and Gale 2003 investigate the optimal design of contests for innovation procurement, and find that the procurer might want to limit the maximum prize available to the most efficient firms – effectively eliminating any positive rents – in order to increase their own expected maximum surplus. The problem of optimal contest design in all-pay contests with spillovers has not been previously analyzed.

This is relevant because principals are constrained in the prizes that they can offer. Many of the tools that principals use to make prizes have spillovers. For example,

if an employer chooses to construct a compensation package using a cash bonus and stock options, then the inclusion of the stock options will generate spillovers. This section analyzes the optimal prize choice when prizes can be constructed from multiple instruments.

Let $\Lambda_i \subset \mathbb{R}^{\tilde{S}_i}$ denote the set of prize functions available to the designer for player i , and let $V : \prod_{i \in I} \tilde{S}_i \times \prod_{i \in I} \Lambda_i \rightarrow \mathbb{R}$ denote the designer's payoff function, i.e., given the pair of scores $\mathbf{s} := (s_1, s_2)$ and the pair of value functions $\mathbf{v} = (v_1(\cdot; \cdot), v_2(\cdot; \cdot))$, $V(\mathbf{s}, \mathbf{v})$ denotes the designer's derived net benefit from the contest.

We make the following (mild) assumptions:

Assumption 1 (Completeness, D1). *For each $i \in I$, set of prizes Λ_i , is convex and its closure contains an element with $v_i(\cdot; \cdot) \equiv 0$.*

Assumption 2 (Productive scores, D2). *For each $i \in I$ and $\mathbf{v} \in \prod_{i \in I} \Lambda_i$, the designer's objective function $V(\mathbf{s}, \mathbf{v})$ is strictly increasing in s_i .*

Assumption 3 (Costly prizes, D3). *For each $i \in I$, $\mathbf{s} \in \prod_{i \in I} \tilde{S}_i$ and $v_{-i} \in \Lambda_{-i}$, $V(\mathbf{s}, \mathbf{v})$ is decreasing in v_i ³⁷.*

The primary complication with the construction in this paper is the mass point is difficult to compute. Fortunately, if the mechanism designer can discriminate between the two players, an optimal mechanism will have no atoms in many specifications. This is formalized in the following proposition.

Proposition 4. *Assume a two-player contest where a fully informed principal with payoff function V chooses the prize $v_i \in \Lambda_i$ for each $i \in I$. Assume that Λ_i and V satisfy assumptions D1 to D3, and that for all i and all $v_i \in \Lambda_i$, assumptions A1 to A4 hold. Then, no contestant in equilibrium can have a positive payoff. Equivalently, no player will have a point-mass as part of their strategy.*

Proposition 4 implies that there will be no strictly dominant player in any discriminating contest design problem where the principal benefits from the efforts of participants and pays for prizes. This proposition comes from the fact that the equilibrium strategy of the dominant player is locally invariant to changes in her prize value. Intuitively, for any contest with a strictly dominant player, there exists a more competitive contest where their prize is reduced and scores are larger.

³⁷That is, if $v_i, \hat{v}_i \in \Lambda_i$ are such that $v_i(s; y) \leq \hat{v}_i(s; y)$ for all $(s, y) \in \tilde{S}_i \times \tilde{S}_{-i}$, then $V(\mathbf{s}, (v_i, v_{-i})) \geq V(\mathbf{s}, (\hat{v}_i, v_{-i}))$.

Proof. Take an optimal choice of $\mathbf{v} := (v_i)_{i \in I} \in \prod_{i \in I} \Lambda_i$. Suppose, by contradiction, that player i has a strictly positive payoff. Her strategy is defined by

$$\tilde{g}_i(s) = \frac{c'_{-i}(s)}{v_{-i}(s; s)} - \int_0^s \frac{v'_{-i}(s; y)}{v_{-i}(s; s)} \tilde{g}_i(y) dy,$$

which does not depend on v_i . Because player $-i$ has an atom, we know that $\tilde{G}_i(\bar{s}) - \tilde{G}_{-i}(\bar{s}) > 0$. Therefore, there exists a $\gamma \in (0, 1)$ such that $\tilde{G}_i(\bar{s}) = \frac{1}{\gamma} \tilde{G}_{-i}(\bar{s})$.

Then, the principal could offer $(\gamma v_i, v_{-i})$ without changing the equilibrium strategy of player i . By the costly prizes Assumption D3, this is weakly preferable given a fixed distribution of s_{-i} .

By construction, player 2's new equilibrium strategy is $\frac{1}{\gamma} \tilde{G}_{-i}(\bar{s})$. This first-order stochastically dominates player $-i$'s original strategy. In fact, it is the same distribution, but with the mass point removed. The productive scores assumption implies that this mechanism is strictly preferred. \square

Proposition 4 demonstrates that the expected welfare of all agents is zero in a large class of contest design problems³⁸. It also suggests the optimality, from a design perspective, of handicapping the most efficient players (as in, the players with lower costs and lower marginal costs) The idea is very much analogous to the conclusion in Che and Gale 2003, for example: handicapping the player that has the technological upper hand causes the less efficient player to become more aggressive, and to choose higher scores than they would otherwise.

Appendix III Numerical Approximation

Iteration method It is possible to approximate the solution by iterating numerically on this sequence:

$$\tilde{g}_{n+1}(s) = \frac{1}{v(s; s)} \left(c'(s) - \int_0^s v'(s; y) \tilde{g}_n(y) dy \right)$$

starting from $\tilde{g}_0 = 0$ to find the true \tilde{g} . There is a much simpler and faster way.

³⁸Which is not to say that there are no settings where it would not apply to. For example, the designer could wish to maximize the agents' expected welfare. In this case, the principal's objective function would violate costly prizes. It would usually also violate productive scores.

Matrix method (1) Consider our original equation

$$\int_0^s v_{-i}(s; y) \tilde{g}_i(y) dy = c(s)$$

and consider this 3×3 discrete approximation of this problem for $s \in [0, 1]$

$$\frac{1}{3} \underbrace{\begin{bmatrix} v_{-i}(1/3, 1/3) & 0 & 0 \\ v_{-i}(2/3, 1/3) & v_{-i}(2/3, 2/3) & 0 \\ v_{-i}(1, 1/3) & v_{-i}(1, 2/3) & v_{-i}(1, 1) \end{bmatrix}}_{\mathbf{V}} \cdot \underbrace{\begin{bmatrix} \tilde{g}_i(1/3) \\ \tilde{g}_i(2/3) \\ \tilde{g}_i(1) \end{bmatrix}}_{\mathbf{g}} \approx \underbrace{\begin{bmatrix} c_{-i}(1/3) \\ c_{-i}(2/3) \\ c_{-i}(1) \end{bmatrix}}_{\mathbf{c}}$$

So, we can approximate $\tilde{g}_i(s)$ with

$$\mathbf{g} = 3\mathbf{V}^{-1}\mathbf{c}$$

Matrix method (2) To get a good estimate, we do the same thing with an $N \times N$ grid for N large on some interval $[0, T]$.³⁹

$$\begin{bmatrix} \tilde{g}_i(1/N) \\ \tilde{g}_i(2/N) \\ \vdots \\ \tilde{g}_i(T) \end{bmatrix} \approx N \begin{bmatrix} v_{-i}(1/N, 1/N) & 0 & \cdots & 0 \\ v_{-i}(2/N, 1/N) & v_{-i}(2/N, 2/N) & \cdots & 0 \\ & & \ddots & \\ v_{-i}(T, 1/N) & v_{-i}(T, 2/N) & \cdots & v_{-i}(T; T) \end{bmatrix}^{-1} \begin{bmatrix} c_{-i}(1/N) \\ c_{-i}(2/N) \\ \vdots \\ c_{-i}(T) \end{bmatrix}$$

Getting the actual strategies Once you get $(\tilde{g}_1, \tilde{g}_2)$ you just have to:⁴⁰

1. take the cumulative sum and divide by N to get $(\tilde{G}_1, \tilde{G}_2)$

$$\mathbf{G1}, \mathbf{G2} = \text{cumsum}(\mathbf{g1})/N, \text{cumsum}(\mathbf{g2})/N$$

2. truncate both distributions at the last value where both are ≤ 1

$$\mathbf{G1}, \mathbf{G2} = \mathbf{G1}[\mathbf{G1} \leq 1 \ \& \ \mathbf{G2} \leq 1], \mathbf{G2}[\mathbf{G1} \leq 1 \ \& \ \mathbf{G2} \leq 1]$$

3. add to each CDF vector so that both end with 1 (add the atom)

$$\mathbf{G1}, \mathbf{G2} = (\mathbf{G1} - \mathbf{G1}[-1] + 1), (\mathbf{G2} - \mathbf{G2}[-1] + 1)$$

³⁹The Python package `allpy` implements the approximation algorithm in this section and computes mixed-strategy equilibria of all-pay contests with spillovers

⁴⁰Sample Python code provided below each item.